DETERMINATION OF THE SUPPORT OF A SYSTEM OF SLIDING VECTORS FUNCTION OF THE RESULTANT MOMENT OF THE VECTOR SYSTEM

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Abstract: The paper presents certain aspects regarding the determination of the support of a sliding vector system resultant function of the resultant moment of the vector system.

Key words: resultant, sliding vector, resultant moment

1. INTRODUCTION

As it is known, a vector is a physical quantity that is defined by three properties: numeric value, direction and sense. The direction of a vector is a line in the space of existence of a vector, which is parallel with the vector. The line that is colinear with the vector, that is, the line that is obtained prolonging the vector beyond the origin and extremity of the vector, is called support line (vector axis).

The versor of the vector axis is the vector the length of which is equal to the measurement unit of the respective vector. The versor specifies a positive sense on the vector axis. If the vector has the same sense as versor, its algebraic value is positive, if it has an opposite sense to the versor, its algebraic sense is negative.

By vector modulus we understand the modulus of the value of its algebraic value, a positive number equal to length of the vector related to the measurement unit.

Replacing the concept of numeric value with modulus, and the concept of direction with support line, one can say that a vector is defined by three properties: modulus, support line and sense.

In classical mechanics, according to the criterion of origin type, the following

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vector categories (classes) are defined: free vectors, the support line of which can take any position in space, parallel with the given direction, application point (origin) not being specified); sliding vectors, the support line of which is fixed in space, and the application point (vector origin) is free on the support line; bound vectors, the support line of which is fixed in space, and the point of application (vector origin)is fixed to the support line.

For any vector $\overline{v}$ the analytical formula in *Oxyz* mark can be written in the form:

$$\overline{v} = v_x\overline{i} + v_y\overline{j} + v_z\overline{k}\ , \tag{1}$$

where v_x, v_y, v_z are the coordinates of the vector.

Coordinates v_x, v_y, v_z are also called scalar components of vector $\overline{v}$ and represent the vector projections on coordinate axes (Fig. 1):

$$v_x = \overline{v}\cdot\overline{i},\ \ v_y = \overline{v}\cdot\overline{j},\ \ v_z = \overline{v}\cdot\overline{k}\ , \tag{2}$$

The vector modulus is given by the equation:

$$\left|\overline{v}\right| = \sqrt{v_x^2 + v_y^2 + v_z^2}\ , \tag{3}$$

2. POLAR MOMENT OF A VECTOR

A $\overline{v}$ sliding vector is considered, with support line (Δ) related to a tri-orthogonal mark *OXYZ*. Polar moment of vector $\overline{v}$ in relation with *O* pole, is called a vector, which is equal to the vectorial product between the position vector of a point on support (Δ), in relation to pole *O* and the given vector. It is noted $\overline{M}_O\left(\overline{v}\right)$:

$$\overline{M}_O\left(\overline{v}\right) = \overline{r}\times\overline{v}\ , \tag{4}$$

where $\overline{r} = \overline{OP},\ \ P\in\left(\Delta\right)$.

The polar moment does not depend on the choice of the point on the support, *P∈ (Δ)* (Fig. 1). Considering another point P_1 *∈ (Δ)*, the following vectorial equation is written between points *O, P*, and P_1 (variation law of coordinates at axes translation):

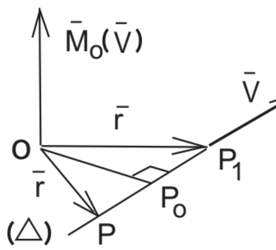

Fig. 1. Polar moment

$$\overline{OP_1} = \overline{OP} + \overline{PP_1},\ \ \overline{r_1} = \overline{r} + \overline{PP_1}\ , \tag{5}$$

We have:

$$\overline{M}'_O\left(\overline{v}\right) = \overline{r_1}\times\overline{v} = \overline{OP_1}\times\overline{v} = \left(\overline{OP} + \overline{PP_1}\right)\times\overline{v} = \overline{OP}\times\overline{v} + \overline{PP_1}\times\overline{v} = \overline{r}\times\overline{v} = \overline{M}_O\left(\overline{v}\right) \tag{6}$$

Since vectors $\overline{PP_1}$ and $\overline{v}$ are colinear, the vectorial product being equal to zero.

We develop $OP_0 \perp (\Delta)$, $P_0 \in (\Delta)$ and we note $b = OP_0$, segment called $\bar{v}$ vector arm.

Polar moment modulus is:

$$\left|\bar{M}_O(\bar{v})\right| = |\bar{r}| \cdot |\bar{v}| \cdot \sin(\bar{r},\bar{v}) = b \cdot |\bar{v}|, \tag{7}$$

Since in the OP_0P rectangular triangle, the equation:

$$b = |\bar{r}| \cdot \sin(\bar{r},\bar{v}), \tag{8}$$

is verified

Equation (8) highlights the fact that the locus of the points in space in relation to which the vector has the same polar moment, is a line parallel with the vector support that includes point *O*.

3. CHARACTERIZATION OF A SLIDING VECTOR

A free vector can be defined with the help of three scalar magnitudes, for instance its projections v_x, v_y, v_z on three axes of Cartesian coordinates.

In addition, in case of a sliding vector $\bar{v}$ it is necessary for the support – line to be known (Δ), along which the vector can travel. If the projections along the axes of vector $\bar{v}$ are known, the directing parameters of the support – line are known. To completely define line (Δ), it is enough for the x_0 and y_0 coordinates for example, of point A_0, in which line (Δ) crosses plane xOy (Fig. 2), to be known; therefore, five independent scalar magnitudes are therefore ultimately necessary, for the characterization of a sliding vector.

To determine a sliding vector, 6 scalar magnitudes are commonly used:

- projections v_x, v_y, v_z on axes of vector $\bar{v}$;
- projections $M_{Ox}(\bar{v})$, $M_{Oy}(\bar{v})$, $M_{Oz}(\bar{v})$ on axes of moments $\bar{M}_O(\bar{v})$ of vector $\bar{v}$ in relation to origin *O* of the axes system

Since, as it has been shown, a sliding vector can be characterized only by three independent scalar magnitudes, it results, that the three scalar magnitudes v_x, v_y, v_z, $M_{Ox}(\bar{v})$, $M_{Oy}(\bar{v})$, $M_{Oz}(\bar{v})$, are not independent, an identically satisfied relation being required to exist among them.

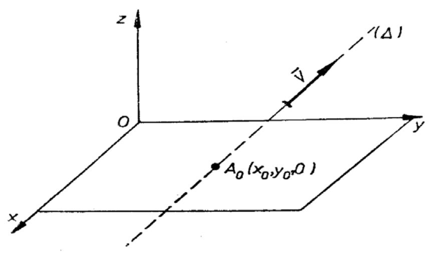

Fig. 2. *(Δ)* crossed with *xOy*

This equation is obtained immediately, if we consider that vectors $\bar{v}$ and $\bar{M}_O(\bar{v})$ are perpendicular, thus the scalar product is null, namely:

$$v_x \cdot M_{Ox}(\bar{v}) + v_y \cdot M_{Oy}(\bar{v}) + v_z \cdot M_{Oz}(\bar{v}) \equiv 0, \tag{9}$$

Identity (9) can be checked directly as well, replacing $M_{Ox}(\bar{v})$, $M_{Oy}(\bar{v})$,

$M_{Oz}(\overline{v})$, with the equations (scalar components of polar moments):

$$M_{Ox}(\overline{v}) = yv_z - zv_y; \quad M_{Oy}(\overline{v}) = zv_x - xv_z; \quad M_{Oz}(\overline{v}) = xv_y - yv_x, \tag{10}$$

and

$$v_x \cdot (yv_z - zv_y) + v_y \cdot (zv_x - xv_z) + v_z \cdot (xv_y - yv_x) \equiv 0,$$

is obtained.

It can be shown that the 6 scalar moments v_x, v_y, v_z, $M_{Ox}(\overline{v})$, $M_{Oy}(\overline{v})$, $M_{Oz}(\overline{v})$, and vectors $\overline{v}$ and $\overline{M}_O(\overline{v})$, respectively, completely characterize a sliding vector.

Indeed, these two vectors cannot be given in any way, they have to be perpendicular to one another. Let thus be two vectors $\overline{v}$ and $\overline{M}_O(\overline{v})$ perpendicular applied in *O*. Normal plane (*P*) is traced on vector $\overline{M}_O(\overline{v})$ and, in this plane, lines (*Δ*) and (*Δ'*) are considered parallel with $\overline{v}$ and situated at distance $d = \dfrac{\left|\overline{M}_O(\overline{v})\right|}{|\overline{v}|}$. One of these lines is the support we seek. In case of Fig. 3, line (*Δ*) is the one that solves the problem. Line (*Δ'*) should be removed, since, if vector $\overline{v}$ would be applied on this line, vector $\overline{M}_O(\overline{v})$ would not comply with the straight drill rule.

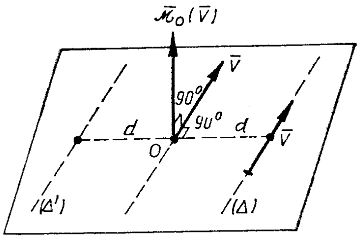

Fig. 3. Line (Δ) support of $\overline{v}$

4. LINE CHARACTERIZATION BY HOMOGENEOUS COORDINATES (PLÜCKER COORDINATES)

Plücker coordinates or homogeneous coordinates (Plücker coordinates introduced by Julius Plücker in the 19th century) are a way of assigning six homogeneous coordinates to each (straight) lines in space. Thus, given a direction (*Δ*) characterized by versor $\overline{u}$ (Fig. 4), Plücker coordinates are defined for direction (*Δ*), by a matrix, the elements of which are:

- components of versor $\overline{u}$ here noted *(a,b,c)*, thus, according to the relation of versor definition, $\overline{u}$ is written:

$$\overline{u} = a\overline{i} + b\overline{j} + c\overline{k}, \quad a = \cos\alpha; \quad b = \cos\beta; \quad c = \cos\gamma, \tag{11}$$

- components of vector $\overline{r} \times \overline{u} = l\overline{i} + m\overline{j} + n\overline{k}$, and according to the definition of the vectorial product, it is written:

$$\overline{r} \times \overline{u} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ x & y & z \\ a & b & c \end{vmatrix} = (yc - zb)\overline{i} + (za - xc)\overline{j} + (xb - ya)\overline{k}, \tag{12}$$

where $\overline{r}$ is the position vector of any point *A* on axis (*Δ*) in relation with the origin of the chosen system of axes, and its formula is: $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$.

After identification results:

$$l = yc - zb; \quad m = za - xc; \quad n = xb - ya \,, \tag{13}$$

Thus, the Plücker coordinates of direction (*Δ)* are written in the form of column matrix:

$$[u] = \begin{bmatrix} a \\ b \\ c \\ yc - zb \\ za - xc \\ xb - ya \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ l \\ m \\ n \end{bmatrix}, \tag{14}$$

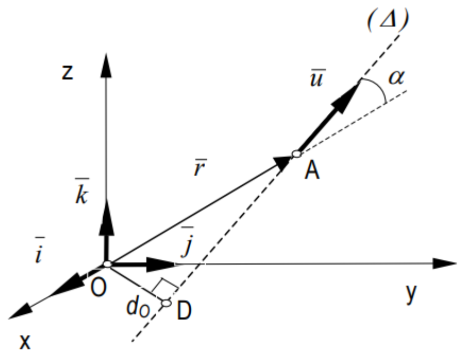

Fig. 4. Line (*Δ*)

Plücker coordinates of a direction have certain obvious properties:

$$\begin{aligned} & a^2 + b^2 + c^2 = 1 \\ & l^2 + m^2 + n^2 = \left|\overline{r} \times \overline{u}\right|^2 = \left|\overline{r}\right|^2 \cdot \left|\overline{u}\right| \cdot \sin^2 \alpha = d_0^2 , \end{aligned} \tag{15}$$

Where $d_0 = OD$ is the distance from point *O* to line (*Δ*) (Fig. 4).

5. DETERMINATION OF A SLIDING VECTOR SUPPORT FUNCTION OF THE POLAR MOMENT

If a sliding vector $\overline{v}$ is given by the analytical formula in *Oxyz* tri-orthogonal landmark,

$$\overline{v} = v_x \overline{i} + v_y \overline{j} + v_z \overline{k} \,, \tag{1'}$$

And its moment is known in relation with the pole *O*, $\overline{M}_O\left(\overline{v}\right)$, that is, if the vector is given by Plücker coordinates, the problem of determining the support of $\overline{v}$ is raised, noted *(Δ)*.

For this, the relation of definition of polar moment is considered (4), where vectors $\overline{v}$ and $\overline{M}_O\left(\overline{v}\right)$ are known, and which will represent a vectorial equation in the unknown vector $\overline{r}$:

$$\overline{M}_O\left(\overline{v}\right) = \overline{r} \times \overline{v} \,, \tag{4'}$$

Vectorial equation ($4'$) is multiplied, vectorial to the left with vector $\overline{v}$:

$$\overline{v} \times \overline{M}_O\left(\overline{v}\right) = \overline{v} \times \left(\overline{r} \times \overline{v}\right), \tag{16}$$

Applying the formula of development of the double vectorial product,

$$\overline{v} \times \overline{M}_O(\overline{v}) = \overline{v}^2 \cdot \overline{r} - (\overline{v} \cdot \overline{r}) \cdot \overline{v}, \tag{17}$$

is obtained, whence:

$$\overline{r} = \frac{\overline{v} \times \overline{M}_O(\overline{v})}{\overline{v}^2} + \left(\frac{\overline{v} \cdot \overline{r}}{|\overline{v}|^2} \right) \overline{v},$$

or:

$$\overline{r} = \frac{\overline{v} \times \overline{M}_O(\overline{v})}{\overline{v}^2} + \left(\frac{\overline{v} \cdot \overline{r}}{|\overline{v}|} \right) \overline{u}, \tag{18}$$

where the versor of the vector $\overline{v}$ was noted

$$\overline{u} = \frac{\overline{v}}{|\overline{v}|}, \tag{19}$$

It is noticed that the first term in (18) represents the position vector in relation with the pole O of point $P_0 \in (\Delta)$, $OP_0 \perp (\Delta)$, where (Δ) is the support line of vector $\overline{v}$:

$$\overline{r}_0 = \frac{\overline{v} \times \overline{M}_O(\overline{v})}{\overline{v}^2}, \quad |\overline{r}_0| = \frac{|\overline{M}_O(\overline{v})|}{|\overline{v}|} = OP_0 = b, \tag{20}$$

where b is the arm of the vector, defined in (8), and the second term in (18) represents the projection of the position vector $\overline{r}$ on the vector support $\overline{v}$:

$$pr_{\overline{v}}\overline{r} = \frac{\overline{v} \cdot \overline{r}}{|\overline{v}|} = \overline{r} \cdot \overline{u}, \tag{21}$$

With these, equation (18) is written in the form of a vectorial equation with an infinity of solutions $\overline{r}$:

$$\overline{r} = \overline{r}_0 + \overline{u} \cdot pr_{\overline{v}}\overline{r}, \quad \overline{r}_0 \perp \overline{u}, \tag{22}$$

its graphic representation being given in Fig. 4.

In Fig. 4, segment P_0P represents the projection of vector $\overline{r}$ on the support of vector $\overline{v}$:

$$P_0P = (pr_{\overline{v}}\overline{r}) \cdot \overline{u}, \tag{23}$$

Noting $\lambda' = pr_{\overline{v}}\overline{r}$, real parameter, equation (23) is written in the form:

$$(\Delta): \overline{r} = \overline{r}_0 + \lambda' \cdot \overline{u}, \quad \lambda' \in \mathbb{R}, \tag{24}$$

and represents the vectorial equation of the $\overline{v}$ .vector support.

Introducing in 924) vector $\overline{r}_0$ and versor $\overline{u}$ (relations (20), respectively (19)), the following equation is obtained for $\overline{v}$ vector support:

$$(\Delta): \overline{r} = \frac{\overline{v} \times \overline{M}_O(\overline{v})}{\overline{v}^2} + \lambda \overline{v}, \quad \lambda \in \mathbb{R}, \tag{25}$$

where λ is a real parameter $(\lambda = \lambda' / |\overline{v}|)$.

Substituting in equation (25) the scalar components of vectors $\overline{v}$ and $\overline{M}_O(\overline{v})$, the following coordinates of an arbitrary point *P* result of the vector support $\overline{v}$:

$$x=\frac{v_y\cdot M_{Oz}\left(\overline{v}\right)-v_z\cdot M_{Oy}\left(\overline{v}\right)}{v_x^2+v_y^2+v_z^2}+\lambda\cdot v_x$$
$$y=\frac{v_z\cdot M_{Ox}\left(\overline{v}\right)-v_x\cdot M_{Oz}\left(\overline{v}\right)}{v_x^2+v_y^2+v_z^2}+\lambda\cdot v_y,\quad \lambda\in\mathbb{R}, \tag{26}$$
$$z=\frac{v_x\cdot M_{Oy}\left(\overline{v}\right)-v_y\cdot M_{Ox}\left(\overline{v}\right)}{v_x^2+v_y^2+v_z^2}+\lambda\cdot v_z$$

where λ is a real parameter, and $M_{Ox}\left(\overline{v}\right)$, $M_{Oy}\left(\overline{v}\right)$, $M_{Oz}\left(\overline{v}\right)$ are given by the equations (scalar components of the polar moment):

$$M_{Ox}\left(\overline{v}\right)=yv_z-zv_y$$
$$M_{Oy}\left(\overline{v}\right)=zv_x-xv_z\ , \tag{27}$$
$$M_{Oz}\left(\overline{v}\right)=xv_y-yv_x$$

and the components of vector $\overline{v}$, by equation:

$$\overline{v}=v_x\overline{i}+v_y\overline{j}+v_z\overline{k}\ , \tag{1''}$$

It is noted that the polar moment is calculated for connected vectors as well:

$$\overline{M}_O\left(\overline{v}\right)=\overline{r}\times\overline{v},\quad \overline{r}=\overline{OP}, \tag{28}$$

in this case point P being the point of application of the connected vector $\overline{v}$.

6. DETERMINATION OF THE RESULTANT SUPPORT DEPENDING ON THE RESULTANT MOMENT

We can see in &3 that in order to be able to determine the support of a vector with the help of the polar moment, the respective vector and its moment have to be perpendicular. This means that in the present case as well, in order to be able to apply the calculation relations for determining the resultant support, function of the resultant moment, these two vectors cannot be given in any way, they have to be perpendicular one upon the other.

The resultant and the resultant moment of a sliding vector system to which the reduction has been made to a point (pole), are reduction elements of the torsor for reducing the system

The scalar product between the resultant of the vector system $\overline{R}$ and the resultant moment of the vector system $\overline{M}$, that is, $\overline{R}\cdot\overline{M}$ (also called the scalar invariant of the vector system, or auto-moment, or the second invariant of the system) is constant.

The set of vectors for which, reduction of the scalar invariant $\overline{R} \cdot \overline{M}$ being made, is null, that is, $\overline{R} \cdot \overline{M} = 0$, forms a particular system of vectors.

Since the scalar invariant is null, it results that vector $\overline{R}$ and $\overline{M}$ are orthogonal, and therefore, the competing, coplanar and parallel vector systems are considered particular force systems. Thus, equations (26) from &5 become:

$$x = \frac{R_y \cdot M_z - R_z \cdot M_y}{R_x^2 + R_y^2 + R_z^2} + \lambda \cdot R_x$$
$$y = \frac{R_z \cdot M_x - R_x \cdot M_z}{R_x^2 + R_y^2 + R_z^2} + \lambda \cdot R_y, \quad \lambda \in \mathbb{R}, \tag{29}$$
$$z = \frac{R_x \cdot M_y - R_y \cdot M_x}{R_x^2 + R_y^2 + R_z^2} + \lambda \cdot R_z$$

where λ is a real parameter, and M_x, M_y, M_z are scalar components (projections on axes) of the resultant moment $\overline{M}$:

$$\overline{M} = M_x \overline{i} + M_y \overline{j} + M_z \overline{k}\ , \tag{30}$$

And the resultant vector components $\overline{R}$, are given by the equation:

$$\overline{R} = R_x \overline{i} + R_y \overline{j} + R_z \overline{k}\ , \tag{31}$$

Function of the particular type (case) of the vector system, the components of the resultant $\overline{R}$ and of the resultant moment $\overline{M}$ will be particularized accordingly as well.

As an example, for determining the support of the resultant of a parallel vector system, we shall consider the following application.

A rigid cube is given in Fig. 5, with side a on which the vector system works (sliding – since the forces that act on a rigid are sliding vectors) forces $F_1=F_2=P$, $F_3=F_4=F_5=2P$.

It is required to determine the support of the system resultant, function of the resultant moment of the vector system.

Scalar components are determined for $\overline{R}$ and $\overline{M}$ and the following table is drawn (Table 1):

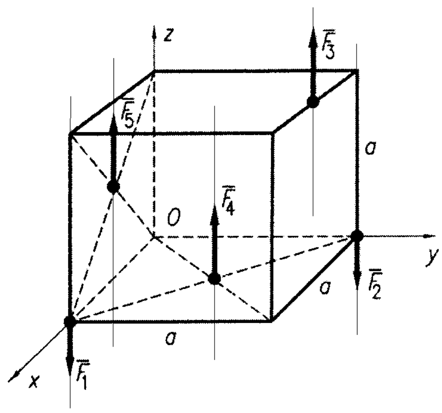

Fig. 5. System of forces

1. Components of $\overline{R}$ și $\overline{M}$

	$F_{x,i}$	$F_{y,i}$	$F_{z,i}$	$M_{Ox,i}$	$M_{Oy,i}$	$M_{Oy,i}$
F_1	0	0	-P	0	Pa	0
F_2	0	0	-P	-Pa	0	0
F_3	0	0	2P	2Pa	-Pa	0
F_4	0	0	2P	Pa	-Pa	0
F_5	0	0	2P	0	-Pa	0
Σ	0	0	4P	2Pa	-2Pa	0

hence:

$$\overline{R} = 4P\overline{k}, \quad \overline{M} = 2Pa\overline{i} - 2Pa\overline{j}\,, \tag{32}$$

Introducing the values obtained in equations (29), we obtain:

$$x = \frac{0\cdot 0 - 4P(-2Pa)}{16P^2} + \lambda\cdot 0, \quad y = \frac{4P\cdot 2Pa - 0\cdot 0}{16P^2} + \lambda\cdot 0, \quad z = \frac{0\cdot(-2Pa) - 0\cdot 2Pa}{16P^2} + \lambda\cdot 4P$$

whence

$$x = \frac{a}{2}, \quad y = \frac{a}{2}, \quad z = 4P\lambda\,, \tag{33}$$

The last relation from equations (33), $z = 4P\lambda$ is an identity, since

$$\lambda = \frac{\overline{R}\cdot r}{R^2} = \frac{4P\overline{k}\cdot\left(x\overline{i} + y\overline{j} + z\overline{k}\right)}{16P^2} = \frac{z}{4P}$$

whence:

$$z = 4P\lambda = 4P\frac{z}{4P}, \Rightarrow z \equiv z$$

Thus, the resultant support is the crossing line of planes $x=a/2$ parallel with yOz and $y=a/2$ parallel with plane xOz.

Fig. 6 represents resultant $\overline{R}$ and its support.

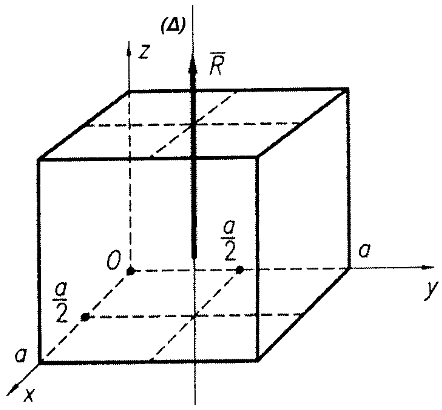

Fig. 6. Resultant support

Result verification is made by determining the central axis of the system with the help of the equation:

$$\frac{M_x - yR_z + zR_y}{R_x} = \frac{M_y - zR_x + xR_z}{R_y} = \frac{M_z - xR_y + yR_x}{R_z}$$

$$\frac{2Pa - y4P + 0}{0} = \frac{-2Pa - z \cdot 0 + x \cdot 4P}{0} = \frac{0 - x \cdot 0 + y \cdot 0}{4P} \Rightarrow$$
$$\Rightarrow \quad 4P\left(2Pa - y4P\right) = 0 \quad \Rightarrow \quad 2a - 4y = 0 \quad \Rightarrow \quad y = a/2 \quad , \qquad (34)$$
$$\Rightarrow \quad 4P\left(-2Pa + x4P\right) = 0 \quad \Rightarrow \quad -2a + 4x = 0 \quad \Rightarrow \quad x = a/2$$

7. CONCLUSIONS

The paper presents aspects regarding determining the support of a sliding vector system resultant, depending on the resultant moment of the vector system, followed by an application example for which the obtained results have been verified.

REFERENCES

[1]. **Bratu, P.,** – *Mecanică teoretică*, Editura IMPULS, București, 2006

[2]. **Voinea, R., Voiculescu, D., Ceaușu, V.,** *Mecanică-ediția a doua revizuită*, E.D.P., București, 1983

[3]. **Mangeron, D., Irimciuc, N.**, *Curs de mecanică rațională cu aplicații în ingineria mecanică, Vol. I, Mecanica solidului rigid, Fasc. 1, Teoria vectorilor alunecători și legați. Cinematica solidului rigid*, Iași, 1973.

[4]. **Corduneanu, E.,** *Mecanică teoretică* I, Universitatea Tehnică ”Gh. Asachi”, Iași, 2018