ASPECTS REGARDING THE APPLICATION IN MECHANICS OF CONCURRENCY CONDITION OF THREE LINES IN A PLANE

RĂZVAN BOGDAN ITU [1], VLAD ALEXANDRU FLOREA[2], DOREL PETRICĂ VITAN [3]

Abstract: The paper presents aspects regarding examples of solving problems of statics of rigid bodies, by application of concurrency condition of three lines in a plane. The examples considered for problems of rigid body equilibrium subject to contact without friction, have been first solved by mechanical considerations, and then, reducing the mechanical problems, they have been solved by geometrical considerations, namely, the problem of concurrency condition of three lines in a plane.

Key words: statics, rigid, concurrency

1. INTRODUCTION

In classical mechanics, the state of equilibrium of a material body can be defined from static or dynamic point of view, both being forms of mechanical equilibrium.

Statics as a division of theoretical mechanics, studies the equilibrium conditions of material systems under the action of applied forces, ignoring motion. The static equilibrium of a material body is characterized by the lack of mechanical motion, that is, there is no modification of its position in time. As opposed to mechanical movement, static equilibrium is equivalent to the state of repose. The state of repose, as well as the state of motion, can be absolute or relative, depending on the reference benchmark. It is recommended for the statics to be approached deductively, starting from the simplest mechanical model (material point), and ending with the system of bodies – the mechanical model of the greatest complexity.

[1] *Lecturer, Eng. Ph.D., University of Petroşani, razvanitu@upet.ro*
[2] *Assoc. Prof., Eng. Ph.D., University of Petroşani, vladalexandruflorea@gmail.com*
[3] *Eng. Ph.D., University of Petroşani, vitandorelpetrica@yahoo.com*

From the point of view of the models of classical mechanics, statistics has two divisions: statistics of the material point and statistics of the solid rigid body, which extends in the case of body systems as well.

Three categories of problems generally occur in statistics, which differ especially by the model of mathematical solving:

a) direct problem, in which the position of equilibrium of the material system in study is supposed to be known, and in the case of its solving, determination of the forces under the action of which the balance is carried out is followed;

b) reverse problem, residing in determining the equilibrium position of the material system analyzed, when the forces acting on it are known;

c) mixed problem, aiming at finding all the unknown elements that refer both to the equilibrium position, and to the forces concurring to it being carried out, when information is known both regarding the equilibrium position, and the forces.

We shall further consider examples of problems of equilibrium of the rigid body, in which the equilibrium position is asked to be found. Concepts of mechanics and geometry are used in solving. The problems in study are considered in a plane and refer to the equilibrium of the rigid body subject to connections without friction.

2. EXAMPLES OF SOLVING FROM A MECHANICAL POINT OF VIEW

In theoretical (classical) mechanics, the *rigid solid body* represents a simplifying theoretical model, specific to this, by which one means a material body in which the distance between two of its points is an (the distance between them is not modified, that is, the solid body is not distorted), irrespective of the magnitudes of the applied forces. Material bodies are not generally found in a free state, they are subject to connections (forms of positioning in space that impose certain restrictions of a geometric order).

As in the case of material points, for the study of the rigid body equilibrium, subject to connections, the connections axiom is used, based on which the connection is suppressed and replaced with corresponding mechanical elements (forces or moments). After all the connections to which a solid rigid body is submitted are suppressed, the following act on it: exterior forces and moments, directly applied; connection forces and moments.

While the reduction torsor in a point O (origin of the reference system) of exterior forces is $\tau_O(\bar{R},\ \bar{M}_O)$, the reduction torsor in O of the connection forces is: $\tau'_O(\bar{R}',\ \bar{M}'_O)$. In this case, the vectorial equilibrium conditions will be:

$$\bar{R} + \bar{R}' = 0,\quad \bar{M}_O + \bar{M}'_O = 0\,, \tag{1}$$

These two vectorial equations are equivalent to those scalars of equilibrium (2), valid when:

$$\bar{R} = X \cdot \bar{i} + Y \cdot \bar{j} + Z \cdot \bar{k};\quad \bar{R}' = X' \cdot \bar{i} + Y' \cdot \bar{j} + Z' \cdot \bar{k}\,;$$

$$\overline{M}_O = M_{Ox} \cdot \overline{i} + M_{Oy} \cdot \overline{j} + M_{Oz} \cdot \overline{k}; \quad \overline{M}_O{}' = M'_{Ox} \cdot \overline{i} + M'_{Oy} \cdot \overline{j} + M'_{Oz} \cdot \overline{k}$$

$$\begin{aligned} &X + X' = 0, \quad Y + Y' = 0, \quad Z + Z' = 0 \\ &M_{Ox} + M'_{Ox} = 0, \quad M_{Oy} + M'_{Oy} = 0, \quad M_{Oz} + M'_{Oz} = 0 \end{aligned}, \tag{2}$$

The calculation algorithm of the reactions is the following:

- the connections of the solid rigid body are suppressed and, in their place, the corresponding reactions are entered. The sense of these reactions is chosen in such a manner, so that in the scalar equilibrium equations, the scalars of the reactions would appear positive:

- a suitable reference system is chosen according to which the static scalar equilibrium equations are written and solved. When from the calculation, the magnitude of a reaction appears positive, that means that the chosen sense is the real one. Otherwise, the respective reaction will have the opposite sense of the initial one.

Momentum equations are written as a rule in relation to the connection which introduces the greatest number of reactions.

Example 1.

A *2l* long homogeneous *AB* bar of *G* weight stays with its ends on two inclined planes, of *α*, respectively *β* angles in relation to the horizontal plane (Fig. 1). The contact is without friction. Determine angle *φ* of the bar with the horizontal plane, for the resting position, as well as the reactions.

Solving.

The bar is freed from connections, introducing the corresponding reactions in supports *A* and *B*, then the scalar equations are written in relation to the reference system chosen in the figure.

$$\begin{aligned} &X \equiv N_A \sin\alpha - N_B \sin\beta = 0 \\ &Y \equiv N_A \cos\alpha - G + N_B \cos\beta = 0 \\ &M_B \equiv -N_A 2l \cos(\alpha + \varphi) + Gl \cos\varphi = 0 \end{aligned}, \tag{3}$$

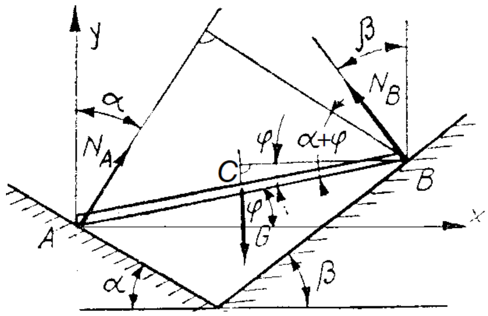

Fig. 1. The bar on the inclined planes

Moving on to solving the system (3), we get:

$$N_B = N_A \frac{\sin\alpha}{\sin\beta}$$

$$N_A \cos\alpha + N_A \frac{\sin\alpha}{\sin\beta} \cos\beta = G \Big| \cdot \sin\beta \;\Rightarrow\; N_A = G \frac{\sin\beta}{\sin(\alpha+\beta)}$$

$$-G \frac{\sin\beta}{\sin(\alpha+\beta)} 2l (\cos\alpha\cos\varphi - \sin\alpha\sin\varphi) + Gl\cos\varphi = 0 \Big| \frac{1}{Gl\cos\varphi} \;\Rightarrow$$

$$\Rightarrow \quad tg\varphi = \frac{\dfrac{2\sin\beta\cos\alpha - \sin(\alpha+\beta)}{\sin(\alpha+\beta)}}{\dfrac{2\sin\alpha\sin\beta}{\sin(\alpha+\beta)}} = \frac{\sin(\beta-\alpha)}{2\sin\alpha\sin\beta}$$

And the reaction in *B* has the value:

$$N_B = G\frac{\sin\alpha}{\sin(\alpha+\beta)},$$

Example 2.

An *ABCD* board, homogeneous, of *G* weight, is suspended by a line, fixed in point *E* to a vertical wall, the *A* peak of the board resting on the vertical plane as well, (Fig. 2).

Knowing that *AB=BC = BE =a*, determine the resting position given by *α* angle, as well as the connecting forces. Friction is ignored.

Solving.

The board being freed from connections, and choosing the reference system as in the figure, the following scalar equilibrium equations are obtained:

$$\begin{aligned} &X \equiv N_A - S\sin\alpha = 0 \\ &Y \equiv S\cos\alpha - G = 0 \\ &M_A \equiv Sa\sin 2\alpha - Ga\frac{\sqrt{2}}{2}\cos\left(\frac{\pi}{4}-\alpha\right) = 0 \end{aligned} \quad , \quad (4)$$

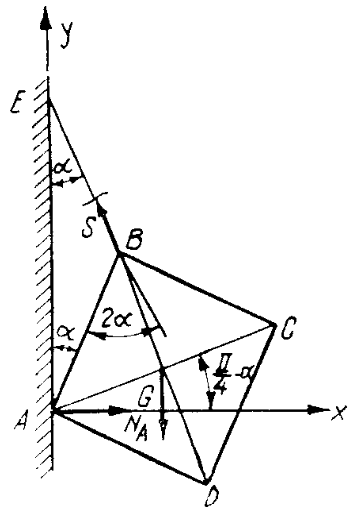

Fig. 2. Homogeneous board

The calculations are made, obtaining:

$$\left.\begin{aligned} &N_A = S\sin\alpha = 0 \\ &S\cos\alpha = G \;\Rightarrow\; S = \frac{G}{\cos\alpha} \end{aligned}\right\} \; N_A = G\frac{\sin\alpha}{\cos\alpha} = Gtg\alpha$$

$$\frac{G}{\cos\alpha}a\sin 2\alpha - Ga\frac{\sqrt{2}}{2}\cos\left(\frac{\pi}{4}-\alpha\right) = 0 \left| \cdot\frac{1}{Ga} \right. \Rightarrow$$

$$\Rightarrow \frac{2\sin\alpha\cos\alpha}{\cos\alpha} - \frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\cos\alpha + \frac{\sqrt{2}}{2}\sin\alpha\right) = 0 \left| \frac{1}{\cos\alpha} \right. \Rightarrow,$$

$$\Rightarrow 2tg\alpha - \frac{1}{2} - \frac{1}{2}tg\alpha = 0 \left| \cdot 2 \right. \Rightarrow \; 4tg\alpha - 1 - tg\alpha = 0$$

$$\Rightarrow \; tg\alpha = \frac{1}{3} \quad respectiv \quad \alpha = arctg\frac{1}{3}$$

Whence, with $tg\alpha = \frac{1}{3}$ the tension in the line is:

$$S = \frac{G}{\cos\alpha} = G\sec\alpha = G\sqrt{1+tg^2\alpha} = G\sqrt{1+\left(\frac{1}{3}\right)^2} = G\sqrt{1+\frac{1}{9}} = G\frac{\sqrt{10}}{3},$$

And the reaction of the vertical wall is:

$$N_A = Gtg\alpha = G\frac{1}{3}.$$

Example 3.

A homogeneous *AB* bar of *G* weight and *2l* long is placed inside a hemisphere of *a* radius (fig. 3). Determine the θ angle of the bar with the horizontal, in its resting position, as well as the reactions, in supports *A* and *D*.

Solving.

The bar is released from the connections, by introducing reactions in supports *A* and *D*. Choosing the *xOy* reference system, the scalar equations of equilibrium are:

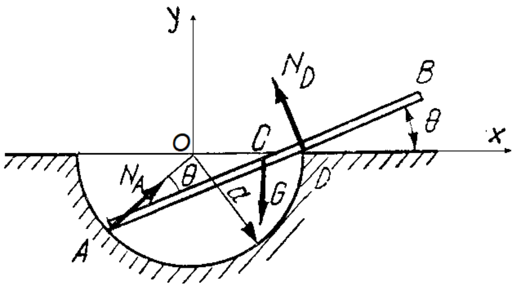

Fig. 3. The bar inside the sphere

$$X \equiv N_A \cos 2\theta - N_D \sin\theta = 0$$
$$Y \equiv N_A \sin 2\theta + N_D \cos\theta - G = 0, \quad (5)$$
$$M_A \equiv N_D 2a\cos\theta - Gl\cos\theta = 0$$

Next we have:

$$\left.\begin{aligned} N_A\cos 2\theta - N_D\sin\theta = 0 \,\middle|\cdot\cos\theta \\ N_A\sin 2\theta + N_D\cos\theta - G = 0 \,\middle|\cdot\sin\theta \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{aligned} N_A\cos 2\theta\cos\theta - N_D\sin\theta\cos\theta = 0 \\ N_A\sin 2\theta\sin\theta + N_D\sin\theta\cos\theta = G\sin\theta \end{aligned}\right\} (+) \Rightarrow$$

$$\Rightarrow N_A\left(\cos 2\theta\cos\theta + \sin 2\theta\sin\theta\right) = G\sin\theta \Rightarrow$$

$$\Rightarrow N_A\cos\left(2\theta - \theta\right) = G\sin\theta \Rightarrow N_A = G\frac{\sin\theta}{\cos\theta} = Gtg\theta$$

$$N_D = N_A\frac{\cos 2\theta}{\sin\theta} = G\frac{\sin\theta}{\cos\theta}\cdot\frac{\cos 2\theta}{\sin\theta} = G\frac{\cos 2\theta}{\cos\theta}$$

$$G\frac{\cos 2\theta}{\cos\theta}2a\cos\theta - Gl\cos\theta = 0 \Rightarrow 2a\cos 2\theta - l\cos\theta = 0 \Rightarrow$$

$$\Rightarrow 2a\left(2\cos^2\theta - 1\right) - l\cos\theta = 0 \Rightarrow$$

$$\Rightarrow\ 4a\cos^2\theta - l\cos\theta - 2a = 0\ \Rightarrow\ \cos\theta = \frac{l \pm \sqrt{l^2 + 32a^2}}{8a}$$

For $\cos\theta = \dfrac{l \pm \sqrt{l^2 + 32a^2}}{8a}$ only the root corresponding to the sign „+" is valid.

The equilibrium is possible for $\cos\theta \le 1$:

$$\frac{l + \sqrt{l^2 + 32a^2}}{8a},$$

whence, making the calculations, condition $l \le 2a$ results, thus the *C* center of gravity of the bar should be at the left of point *D*.

The reactions are:

$$N_A = Gtg\theta,\ \ N_D = G\frac{l}{2a}\ \ (\mathit{from}\ \ M_A = 0,\ \ \mathit{the}\ \ \mathit{system}\ \ (5)),$$

Example 4.

The *AB* homogeneous bar, of *G* weight and *2l* long leans in *A* against a vertical wall, and in *D* against the edge of another wall, located at distance *a* from the first one (Fig. 4). The contact is without friction. Determine the angle θ of the bar with the horizontal plane in its resting position, similarly the reactions in supports *A* and *D*.

Solving.

The bar is released from connections, by introducing reactions in supports *A* and *D*, each reaction having as normal direction at the surface which does not have singular point in the contact point.

In order to obtain projection equations as simple as possible, the *xAy* reference system is chosen, so that as many as possible forces would be projected in real size.

Scalar equilibrium equations are:

$$\begin{aligned} & X \equiv N_A - N_D\sin\theta = 0 \\ & Y \equiv N_D\cos\theta - G = 0 \\ & M_A \equiv N_D\frac{a}{\cos\theta} - Gl\cos\theta = 0 \end{aligned}\quad,\ (6)$$

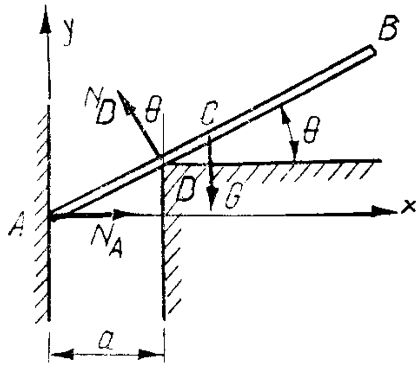

Fig. 4. Homogeneous bar

We move to the solving of the system (6):

$$\left.\begin{aligned} & N_A - N_D\sin\theta = 0 \\ & N_D\cos\theta - G = 0\ \Rightarrow\ N_D = \frac{G}{\cos\theta} \end{aligned}\right\}\ \Rightarrow\ N_A = G\frac{\sin\theta}{\cos\theta} = Gtg\theta$$

$$\frac{G}{\cos\theta}\cdot\frac{a}{\cos\theta} - Gl\cos\theta = 0\,\bigg|\cdot\frac{\cos^2\theta}{G}\ \Rightarrow\ \cos^3\theta = \frac{a}{l}\ \Rightarrow\ \cos\theta = \sqrt[3]{\frac{a}{l}}$$

For the equilibrium to be possible:

$$\sqrt[3]{\frac{a}{l}} \le 1, \quad deci \quad a \le l\,,$$

Reactions:

$$N_A = G\sqrt{\left(\frac{l}{a}\right)^{\frac{2}{3}} - 1}, \quad N_D = G\sqrt[3]{\frac{l}{a}}$$

In the applications presented, the angles made by rigid bodies at rest have been determined by mechanical considerations. In the following, the angles will be determined reducing the problem of mechanics to the problem of concurrency condition of three lines in a plane.

3. CONCURRENCY CONDITION OF THREE LINES IN A PLANE

Out of the principal methods of demonstrating concurrency of minimum three lines in a plane, we mention the following:

1) using the definition of concurrent lines: for lines *a, b, c* it is demonstrated that $a \cap b = \{P\}$, $a \cap c = \{Q\}$ and that $P = Q$ or, analogously, it is demonstrated that $a \cap b = \{P\}$ and that $P \in c$.In the applications presented, the determination of the angles made by rigid bodies at rest were determined by mechanical considerations. In the following angles will be determined by reducing the mechanical problem to the problem of the condition of competition of three straight lines in the plane.

2) using the collinearity of certain points: for lines *a, b, c* it is demonstrated that $a \cap b = \{P\}$ and □either $Q, R \in c$. If P, Q, R are colinear, then *a, b, c* are concurrent in *P*.

3) using the converse of Ceva theorem.

4) using the concurrency of important lines in a triangle: we identify a triangle where the given lines become medians, or bisectors, or heights or mediators.

5) analytical method.

6) vectorial method.

7) using the converse of Carnot theorem.

To determine the angles in the case of rigid bodies taken as example, in the following, we consider that the support lines of the forces (directly applied and of connection) acting on the rigid body, should be concurrent. The equations of the support lines will be explained as the equation of line determined by appoint and a direction (given slope).

We thus have for *example* 1 (Fig. 1), the equations of these lines (Fig. 5):

$$(Au): y = xctg\alpha\,, \tag{7}$$

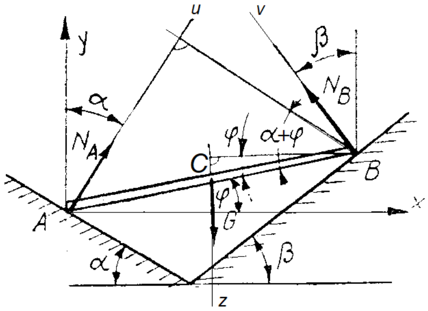

Fig. 5. Support lines, example 1

$$(Bv): y-2l\sin\varphi=-ctg\beta\left(x-2l\cos\varphi\right), \tag{8}$$

$$(Cz): x=l\cos\varphi, \tag{9}$$

We stipulate that the three lines would be concurrent.

We remove y from the first equations:

$$xctg\alpha-2l\sin\varphi=-ctg\beta\left(x-2l\cos\varphi\right), \tag{10}$$

We substitute (10) in x with its value of (9), and we get the condition that is looked for.:

$$ctg\varphi\cdot ctg\alpha-2\sin\varphi=-ctg\beta\left(\cos\varphi-2\cos\varphi\right), \tag{11}$$

or:

$$ctg\alpha-2tg\varphi=ctg\beta \;\Rightarrow\; tg\varphi=\frac{ctg\alpha-ctg\beta}{2}=\frac{\sin\left(\beta-\alpha\right)}{2\sin\alpha\sin\beta}, \tag{12}$$

For example 2 (Fig. 2) the equations of the lines (Fig. 6) are:

$$(Au): y=0, \tag{13}$$

$$\begin{aligned}&(Bv): y-l\cos\alpha=-tg\left(90^0+\alpha\right)\left(x-l\sin\alpha\right) \;\Rightarrow\\ &\Rightarrow\; y=-\left(x-l\sin\alpha\right)ctg\alpha+l\cos\alpha\end{aligned} \tag{14}$$

$$(Oz): x=\frac{l\sqrt{2}}{2}\cos\left(\frac{\pi}{4}-\alpha\right), \tag{15}$$

We stipulate the condition that the three lines are concurrent.

We remove y from the first equations, and making the calculations:

$$\begin{aligned}&-\left(x-l\sin\alpha\right)ctg\alpha=-l\cos\alpha \;\Rightarrow\\ &\Rightarrow\; \left(x-l\sin\alpha\right)ctg\alpha=l\cos\alpha\,\Big|\cdot\frac{1}{\cos\alpha}\\ &\Rightarrow\; \frac{x}{\sin\alpha}-l=l\Rightarrow\; x=2l\sin\alpha\end{aligned} \tag{16}$$

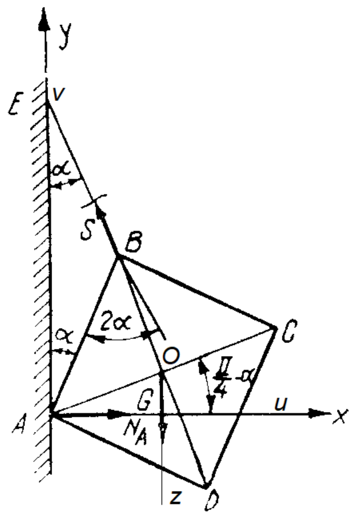

Fig. 6. The board, example 2

We substitute (16) pe x with its value in (15), and we get the condition looked for:

$$\begin{aligned}
&2l\sin\alpha = l\frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{2}\cos\alpha + \sin\frac{\pi}{2}\sin\alpha\right) \Rightarrow \\
&\Rightarrow \ 2\sin\alpha = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}\left(\cos\alpha + \sin\alpha\right) \ \Rightarrow \\
&\Rightarrow \ 2\sin\alpha = \frac{1}{2}\left(\cos\alpha + \sin\alpha\right)\big|\cdot 2 \Rightarrow \ 4\sin\alpha = \cos\alpha + \sin\alpha \ \Rightarrow \\
&\Rightarrow \ 3\sin\alpha = \cos\alpha \ \Rightarrow \ 3tg\alpha = 1 \ \Rightarrow \ tg\alpha = \frac{1}{3}
\end{aligned}, \qquad (17)$$

For example 3 (Fig. 3), the equations of the lines (Fig. 7) are:

$$(Au): y = xtg2\theta, \qquad (18)$$

$$(Dv): y = -(x-a)ctg\theta, \qquad (19)$$

$$(Cz): x = l\cos\theta - a\cos 2\theta, \qquad (20)$$

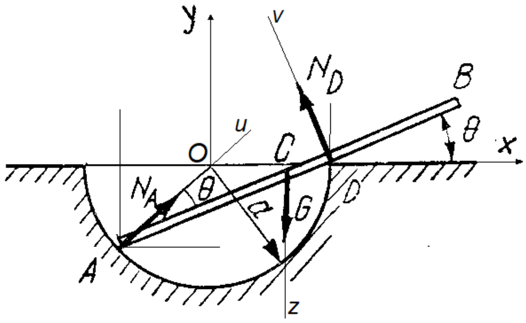

Fig. 7. The bar from example 3

We stipulate that the three lines are concurrent.

We remove y from the first equation, making the calculations:

$$xtg2\theta = -(x-a)ctg\theta \quad sau \quad x\left(tg2\theta + ctg\theta\right) = actg\theta, \qquad (21)$$

Substituting x with its equation from (20), we get:

$$l\cos\theta = a\cos 2\theta \quad sau \quad 2a\cos^2\theta - l\cos\theta - a = 0, \qquad (22)$$

It results:

$$\cos\theta = \frac{l + \sqrt{l^2 + 32a^2}}{8a}, \quad de \quad unde \quad l + \sqrt{l^2 + 32a^2} \triangleleft 8a \ \Rightarrow l \triangleleft 2a, \qquad (23)$$

For example 4 (fig. 4) the equations of the lines (Fig. 8) are:

$$(Ax): y = 0, \qquad (24)$$

$$(Du): y - atg\theta = tg\left(90^0 + \theta\right)(x-a) \ \Rightarrow \ y = -(x-a)ctg\theta + atg\theta, \qquad (25)$$

$$(Cz): x = l\cos\theta, \qquad (26)$$

We stipulate that these lines are concurrent.

We remove y from the first equations and the following calculations are made:

$$-(x-a)ctg\theta + atg\theta \quad sau \quad (x-a)ctg\theta = atg\theta \quad \Rightarrow$$
$$\Rightarrow \quad x = a\frac{tg\theta + ctg\theta}{ctg\theta} \quad , \qquad (27)$$

We substitute x with its equation of (26), and we get:

$$l\cos\theta = a\frac{tg\theta + ctg\theta}{ctg\theta} \quad \Rightarrow$$
$$\Rightarrow \quad l\cos\theta\frac{\cos\theta}{\sin\theta} = a\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right) \quad \Rightarrow$$
$$\Rightarrow \quad l\frac{\cos^2\theta}{\sin\theta} = a\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}\Big|\cdot\sin\theta \quad , \qquad (28)$$
$$\Rightarrow \quad l\cos^2\theta = \frac{a}{\cos\theta} \quad \Rightarrow \quad \cos^3\theta = \frac{a}{l} \quad \Rightarrow$$
$$\Rightarrow \quad \cos\theta = \sqrt[3]{\frac{a}{l}}$$

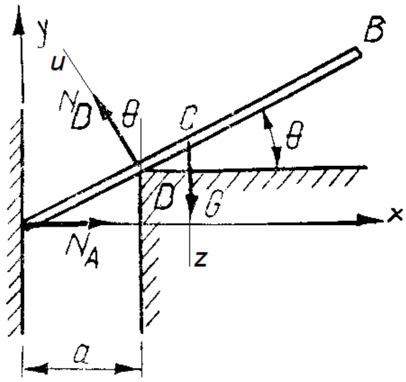

Fig. 8. The bar for example 4

4. CONCLUSIONS

The paper presents aspects regarding certain examples pf solving problems of statics of rigid bodies, by applying concurrency condition of three lines in a plane. As examples, problems of rigid bodies equilibrium have been considered, subject to connections without friction, first solved by mechanical considerations, and then, reducing the problems of mechanics, by geometrical considerations. By the problem of concurrency condition of three lines, angles have been determined, in the case of the rigid body equilibrium position, subject to connections. To be noticed that the direct applied forces and of connection, should be together a number of three, in order to be able to apply the condition of concurrency of three lines in a plane.

REFERENCES

[1]. Coșniță, C., Sager, I., Matei, I., Dragotă, I. *Culegere de probleme de geometrie analitică*, E.D.P., București,1963.

[2]. Marc, B.I. *Elemente de mecanică. Note de curs*, Editura UNIVERSITAS, Petroșani, 2023

[3]. Sarian, M., ș.a. *Probleme de mecanică – pentru ingineri și subingineri*, E.D.P., București, 1975

[4]. *** *https://www.math.uaic.ro/~oanacon/depozit/Tema12.pdf*