ASPECTS REGARDING THE METHOD OF INVERSE MOTION AND COLLINEARITY IN MECHANICS

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Abstract: Comprehension, acquiring and correct application of a method of calculation in solving a problem is a matter of experience, which is gained based on solving a considerable number of problems. The paper presents aspects regarding the application of the problem of inverse motion and of the property of mechanical collinearity.

Key words: kinematics, inverse motion, collinearity

1. INTRODUCTION

Kinematics, as part of mechanics, studies mechanical motion, disregarding mass, forces and moments, that is, considering exclusively its geometric aspect.

The instrument of solving any problem of mechanics is the application of one or several equations (in simple cases), or of one or several methods (in more complex cases).

One should point out that the comprehension, acquisition and correct application of a method of calculus, in the solving of a problem, is a question of experience, which is gained based on solving a considerable number of problems.

The thorough knowledge of solving methods leads to a rapid and safe solutioning of the problems, regardless of their degree of difficulty.

One of the methods of solving the problems of mechanics is the method of inverse motion as well.

Similarly, in mechanics, particularly in kinematics, as well as in geometry, there are aspects in which reference is made to the collinearity property, both in theory and in applications.

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In the following, aspects on the method of inverse motion and collinearity property will be presented, with an example from an applicative problem.

2. METHOD OF INVERSE MOTION

As a principle and field of applicability, the method of inverse motion is applied in those problems in which several bodies are in motion, and the motion of a mobile body compared to another body is searched for (the case of relative motion).

In essence, it consists of the observation of the kinematic phenomena, which take place in the respective problem, using the motion of one of the mobile bodies in the problem.

The notions used within this method are based on the use spatial coordinates (the Cartesian coordinate from analytical geometry) and of the temporal coordinate (time) instead of space (the route covered) and of time (time interval), used in the solving of several problems of kinematics,

In the equations of this method the following notions intervene:

$x = x(t)$ – linear coordinate (abscise) of the mobile point that is in rectilinear motion on *Ox* axis;

$\theta = \theta(t)$ – angular coordinate of a mobile point in circular motion or of a solid body in rotation;

t – time coordinate or the time indicated by the clock at a certain moment;

v – linear velocity of a mobile point in rectilinear rotation motion on *Ox* axis;

ω – angular velocity of the mobile point in circular motion of a solid body in rotation motion;

a – liner acceleration of a mobile point in rectilinear motion on *Ox* axis;

ε – angular acceleration of a mobile point in circular motion or of a solid rigid in rotation.

Referring to the equations of the method, by motion equation the expression of its linear or angular coordinate and of velocity at a certain moment being understood, within the method of inverse motion, one can sooner talk about a procedure than of equations, since the equations that are written as a result of the application of this method, are simple and well-known. The basic procedure of this method lies in giving all the bodies that are in motion in a problem, an inverse motion as to one of them, compared with which the study of the motion of the others are required.

Sometimes, it is not even necessary to ask for this, but by applying this method, the respective problem can be reformulated in a much simpler way.

The stages of the method are the following:

a) identify the bodies with which we are dealing, be it for example C_1 and C_2, having in this moment velocities v_1 and v_2;

b) choose between these two bodies the one that stays fixed (can be any of those);

If we choose body C_1 that stays fixed, (compared to which the movement of body C_2 should be determined, then both bodies are given velocity $-\overline{v}_1$. As a result, body C_1 will stay on the same spot (at rest), and body C_2 will get velocity $\overline{v}_2' = \overline{v}_2 - \overline{v}_1$.

When C_2 is the body that should stay fixed (compared to which the motion of body C_1) should be determined, then both bodies are given velocity $-\overline{v}_2$. As a result, body C_2 stays fixed, and body C_1 gets velocity $\overline{v}_1' = \overline{v}_1 - \overline{v}_2$.

It is easily noticed that $\overline{v}_2' = -\overline{v}_1'$, that is, the relative motion of body C_2 as to body C_1 is inverse, compared to the relative motion of C_1 as to C_2; this observation will help in solving certain problems as well.

After the operations (stages) have been done so far, the problem changed into another, much simpler one; in this stage we either reformulate the problem initially given, or we directly make the calculations required in the problem.

This problem (methodology) can be generalized for the case in which in the respective problem there are more than two mobile bodies.

As indications and specification, in the application of this method, we should consider the following:

- the presented procedure can be applied to determine both velocities and accelerations, although it has been presented only for velocities;

- all vectors (velocities and accelerations) are summed up only vectorially; for calculations, the analytical method of decomposition on the axes of a reference system can be used.

An example of application of this method will be presented in the following.

3. COLLINEARITY PROPERTY

In geometry, collinearity is the property of a number greater than two points to belong to the same line.

Several non-collinear points are points that cannot belong to the same line. This can be also demonstrated using vectors and complex numbers, as for coplanarity.

Out of the methods specific to demonstrate collinearity used in geometry, we mention the following methods.

Demonstration of collinearity with the help of elongated angle (additional angles).

If A and C are situated on either side of line BD, and $m(<ABD)+m(<DBC)=180^0$, then points A, B and C are collinear (Fig. 1).

Demonstration of collinearity using the reciprocal of the theorem of opposite angles at the apex.

If point B is situated on line DE, and A and C are on either side of line DE, and <ABD = <CBE, then points A,B,C are collinear (Fig. 2).

Demonstration of collinearity by identifying a line that includes the respective points.

To show that points A,B,C are collinear, a line is identified to which they should belong.

The condition of collinearity of three points, $A(x_1,y_1)$, $B(x_2,y_2)$, $C(x_3,y_3)$ is *obtained if we stipulate that $C(x_3,y_3)$ point would verify the equation of line AB, that is,:*

$$\frac{y_3 - y_1}{y_2 - y_1} = \frac{x_3 - x_1}{x_2 - x_1}, \tag{1}$$

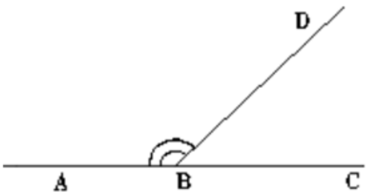

Fig. 1. Additional angles

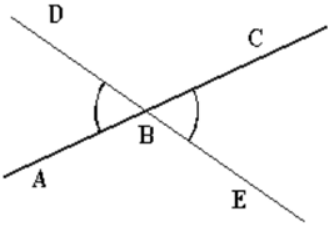

Fig. 2. Angles opposed at the peak

The condition of collinearity of the three points can also be written in the form of a determinant:

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}, \tag{2}$$

Demonstration of point collinearity using vectors.

Two vectors are collinear if they have the same direction. This happens when both vectors are nonnull and their supporting lines are parallel or coincide, in the case when one of the vectors is null. The parallelism of the vectors represents a particular case of their collinearity, which is explicable by the fact that free vectors have no fixed position and can be translated in any point of the plane.

Demonstration of collinearity of points using the applications of complex numbers in geometry

If points A, *B*, C have respectively z_A, z_B, z_C affixes, then *A*, *B*, *C* are collinear if and only if $(z_B - z_A)/(z_C - z_A) \in \mathbb{R}^*$.

4. APPLICATION OF THE METHOD OF INVERSE MOTION AND COLLLINEARITY CONDITION IN THREE POINTS ALONG THE SAME AXES

An application will be presented below, in which the method of inverse motion is used and the collinearity condition applied in three points that move along different trajectories and are obligated to permanently be on the same line.

The following problem is considered: Three boats move on still water, on parallel trajectories, the distances between them *a = 20 m* and *b = 6 m*. Boat *A* moves with constant speed v_1 and boats *B* and *C* move in opposite direction with constant speeds v_2 and v_3. Knowing the speeds $v_1 = 5\ m/s$ and $v_2 = 4\ m/s$, v_3 has to be found out so that the boats would permanently be in a straight line. (Fig. 3, a).

We actually intend to find the equation that should exist between the values of the initial speeds, so that all along the movement, the three bodies would stay collinear.

In order to solve the collinearity problem, we should admit the following two simplifying hypotheses: ignore water resistance and consider the three bodies as material points.

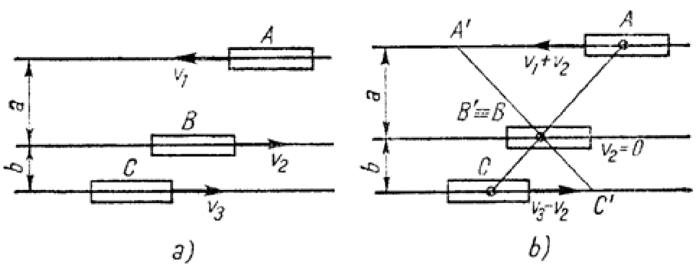

Fig. 3. Motion of the boats and the first variant

4.1. Application of the method of inverse movement

Applying the method of inverse motion, the entire system is given a translation motion with a velocity equal to, and in opposite sense with the speed of one of the boats, for example we consider the speed of boat *B* (Fig. 3, b). The solver can convince himself that it is irrelevant which of the inverse motions is chosen, solving the problem in the other two variants.

1) *First variant*. In this case, boat *B* stays in the same place, because it is given velocity $v_2 - v_2 = 0$, and the other two boats are given the velocities:

- boat *A*: $v_1 + v_2$;

- boat *C*: $v_3 - v_2$, supposed to be oriented in the initial sense of v_3, the hypothesis being $v_3 > v_2$, which we acquiesce.

After some time, *t*, boat *A* will reach position *A'*, boat *C* will reach position *C'*, and boat *B* will stay in the same place (*B'* = *B*), the following spaces being covered, respectively:

$$AA' = (v_1 + v_2)t, \quad BB' = 0, \quad CC' = (v_3 - v_2)t\,, \tag{3}$$

Noticing the similarity of triangles *BAA'* and *BCC'* one can write:

$$\frac{AA'}{CC'} = \frac{a}{b} = \frac{(v_1 + v_2)t}{(v_3 - v_2)t} = \frac{v_1 + v_2}{v_3 - v_2}, \tag{4}$$

From the second and from the last relation, the velocity values results:

$$v_3 = \frac{bv_1 + (a+b)v_2}{a} = \frac{6 \cdot 5 + 26 \cdot 4}{20} = 6{,}7m/s, \tag{5}$$

2) Second variant. The entire system is given a translation motion with a velocity equal to and inverse with the speed of boat *A* (fig. 4)

In this situation, boat *A* stays in the same place, because it is given velocity $v_1 - v_1 = 0$, and the other two boats are given velocities:

- boat *B*: $v_1 + v_2$;
- boat *C*: $v_3 + v_1$, supposed to be oriented in the

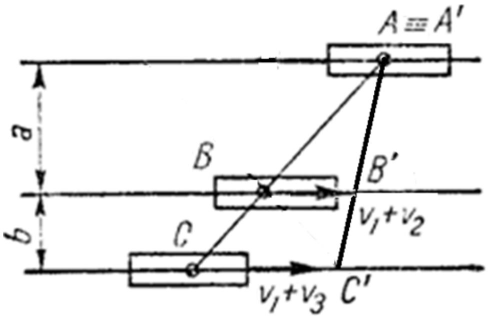

Fig. 4. Second variant

initial sense of v_3 the hypothesis being $v_3 > v_1$, which we admit.

After some time, *t,* boat *B* will reach position *B'*, boat *C* will reach position *C'*, and boat *A* will stay in the same place (*A'* = *A*), the following spaces being covered, respectively:

$$AA' = 0, \quad BB' = (v_2 + v_1)t, \quad CC' = (v_3 + v_1)t, \tag{6}$$

Noticing the similarity of triangles *BAA'* and *BCC'* one can write:

$$\frac{BB'}{CC'} = \frac{a}{a+b} = \frac{(v_2 + v_1)t}{(v_3 + v_1)t} = \frac{v_2 + v_1}{v_3 + v_2}, \tag{7}$$

From the second relation and from the last the velocity value v_3 results:

$$v_3 = \frac{bv_1 + (a+b)v_2}{a} = \frac{6 \cdot 5 + 26 \cdot 4}{20} = 6{,}7m/s, \tag{5'}$$

3) Third variant. The entire system is given a translation motion, with a velocity equal to and of opposite direction with the speed of boat C (Fig. 5).

In this situation boat *C* stays in the same place, since it gets velocity $v_3 - v_3 = 0$, and the other two boats get velocities:

- boat *A*: $v_1 + v_3$;

- boat *B*: $v_2 - v_3$, supposed to be oriented in the initial sense of v_2 the hypothesis being $v_3 > v_2$, which we admit.

After some time, *t,* boat *A* will reach to position *A'*, boat *B* to position *B'*, and boat *C* will stay in the same place ($C' = C$), covering the spaces, respectively:

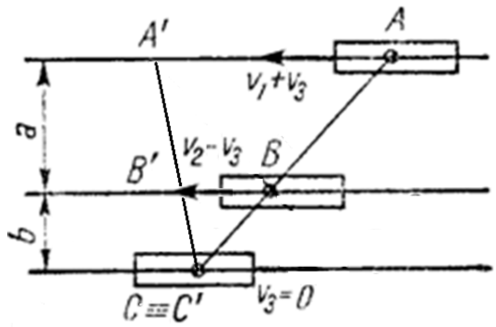

Fig. 5. Third variant

$$AA' = -(v_1 + v_3)t, \quad BB' = (v_2 - v_3)t, \quad CC' = 0, \tag{8}$$

Noticing the similarity of triangles *BAA'* and *BCC'* one can write:

$$\frac{BB'}{AA'} = \frac{a}{a+b} = \frac{(v_2 - v_3)t}{-(v_3 + v_1)t} = -\frac{v_2 - v_3}{v_1 + v_3}, \tag{9}$$

From the second relation and from the last, the value of v_3 velocity results:

$$v_3 = \frac{bv_1 + (a+b)v_2}{a} = \frac{6 \cdot 5 + 26 \cdot 4}{20} = 6,7 m/s, \tag{5''}$$

4.2. Application of collinearity condition

We shall further present the application in which collinearity condition is used, applied in three points that move along different trajectories, and are obliged to be permanently on the same line.

The problem is considered in the following formulation: three boats travel in three parallel directions, the distance between them being *a = 400* m and *b = 200 m*.

Knowing that the speeds of two of the boats are = V_1 = *12 km/h* and V_2 = *16 km/h* (having the sense shown in figure 6), let us calculate the speed of the third boat, so that between the three boats there would always be a straight line(a relation between the boats so that the boats during their motion would be collinear).

Similarly, to solve the problem, we shall admit the following two simplifying hypotheses: ignore water resistance and consider the three bodies (boats) as material points.

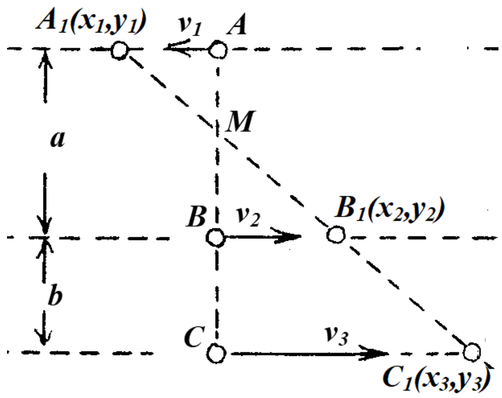

Fig. 6. Motion of the boats

In order to establish collinearity condition of the three material points, (Fig. 6), $A_1(x_1, y_1)$, $B_1(x_2, y_2)$ and $C_1(x_3, y_3)$, we turn at the beginning to our knowledge of analytical geometry,

Collinearity condition lies in the equation from below, according to (2):

$$\delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}, \tag{10}$$

To examine the coordinates of points A', B' and C' at a moment $t > 0$, calculated from the moment of launching the bodies, we turn to our knowledge of kinematics from mechanics:

$$\begin{aligned} x_1 &= -v_1 t, \quad y_1 = a + b \\ x_2 &= v_2 t, \quad y_2 = b \\ x_3 &= v_3 t, \quad y_3 = 0 \end{aligned} \quad , \tag{11}$$

Substituting then (17), in determinant (16), we get:

$$\delta = \begin{vmatrix} -v_1 t & a+b & 1 \\ v_2 t & b & 1 \\ v_3 t & 0 & 1 \end{vmatrix}, \tag{12}$$

So that by developing this determinant equal to zero we get:

$$v_3 t(a+b) - 0 \cdot (-v_1 t - v_2 t) + 1 \cdot \left[-v_1 t b - v_2 (a+b)\right] = 0,$$

After reducing similar terms [2] and dividing by ($t > 0$), we get the following result in the end:

$$v_3 a = v_2 \left(a + b\right) + v_1 b \,, \tag{13}$$

We shall further prove collinearity of points, using this time the vectors (Fig. 7).

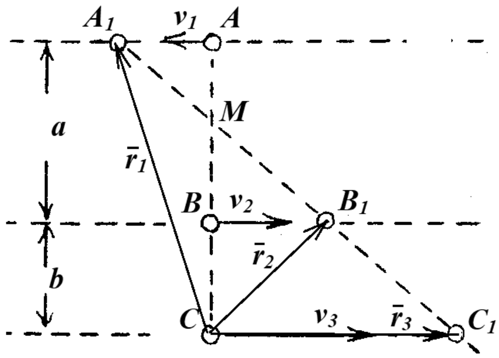

Fig. 7. Position vectors of points

Points A_1, B_1, C_1 are collinear if and only if vectors $\overline{A_1B_1}$ and $\overline{A_1C_1}$ are collinear, that is, if and only if there is $\alpha \in R$ so that:

$$\overline{A_1C_1} = \alpha \overline{A_1B_1} \quad sau \quad \frac{x_3 - x_1}{x_2 - x_1} = \frac{y_3 - y_1}{y_2 - y_1}, \tag{14}$$

It results, considering (11):

$$\frac{v_3 t - \left(-v_1 t\right)}{v_2 t - \left(-v_1 t\right)} = \frac{0 - \left(a + b\right)}{a - \left(a + b\right)} \;\Rightarrow\; v_3 a = v_2 \left(a + b\right) + v_1 b \,, \tag{15}$$

It is noticed, that equation (15) is identical to equation (13).

Next, we shall determine collinearity relation for points A_1, B_1 and C_1 using complex numbers.

Associating $z = x + iy$, $M(x,y)$, to R set of real numbers, axis Ox corresponds, called in this context real axis, ant to iR set of imaginary numbers, Oy axis corresponds, called imaginary axis.

The plane the points of which are identified to the complex numbers by $g \circ f$ function, defined above, is called complex plane.

The affixes of A_1, B_1 and C_1 points, according to Fig. 8, are z_1, z_2, z_3.

Points $A_1(z_1)$, $B_1(z_2)$ and $C_1(z_3)$ are in collinearity relation, if, and only if $(z_3-z_1)/(z_3-z_2)$ belongs to R.

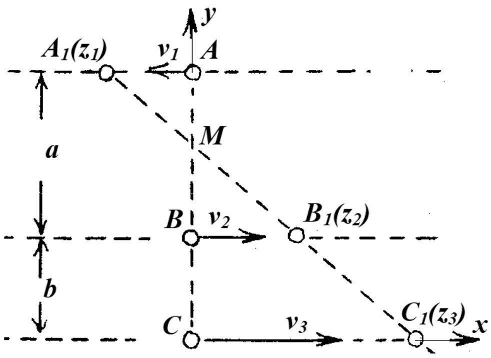

Fig. 8. Affixes of points A_1, B_1 and C_1

According to Fig. 8 and the kinematic equations of mechanics, we have

$$z_1 = x_1 + iy_1,\ \ z_1 = -v_1 t + i(a+b)$$

$$z_2 = x_2 + iy_2,\ \ z_2 = v_2 t + ib \qquad , \tag{16}$$

$$z_3 = x_3 + iy_3,\ \ z_3 = v_3 t + i \cdot 0$$

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{x_3 + iy_3 - (x_1 + iy_1)}{x_2 + iy_2 - (x_1 + iy_1)} = \frac{x_3 - x_1 + i(y_3 - y_1)}{x_2 - x_1 + i(y_2 - y_1)} = \lambda , \tag{17}$$

It results:

$$\frac{v_3 t - (-v_1 t) + i\left[0 - (a+b)\right]}{v_2 t - (-v_1 t) + i\left[b - (a+b)\right]} = \frac{v_3 t + v_1 t - i(a+b)}{v_2 t + v_1 t - ia} = \lambda , \tag{18}$$

where $\lambda \in \mathbb{R}^*$.

Thus, we shall have:

$$\frac{v_3 t + v_1 t - i(a+b)}{v_2 t + v_1 t - ia} = \lambda \ \Leftrightarrow \ (v_3 + v_1)t - \lambda(v_2 + v_1)t - i\left[(a+b) - \lambda a\right] = 0, \tag{19}$$

For the last relation in (19) to be $(v_3 + v_1)t - \lambda(v_2 + v_1)t - i\left[(a+b) - \lambda a\right] = 0$, we must have: $(v_3 + v_1)t - \lambda(v_2 + v_1)t = 0$ and $i\left[(a+b) - \lambda a\right] = 0$, whence the following will result, respectively:

$$\lambda = \frac{v_3 + v_1}{v_2 + v_1} \ \text{ and } \lambda = \frac{a+b}{a}, \tag{20}$$

Equaling the two equations in (20), it results:

$$\frac{v_3 + v_1}{v_2 + v_1} = \frac{a+b}{a} \quad \Rightarrow v_3 a = v_2\left(a+b\right) + v_1 b \,, \tag{21}$$

Equation (21) is identical to equation (13).

Next, we shall consider the demonstration of collinearity with the help of the elongated angle (additional angles).

If A_1 and C_1 are situated on one side and the other of line CB_1 and $m(<A_1B_1C) + m(<CB_1C_1)=180^0$ (Fig. 9), then points A_1, B_1 and C_1 are collinear.

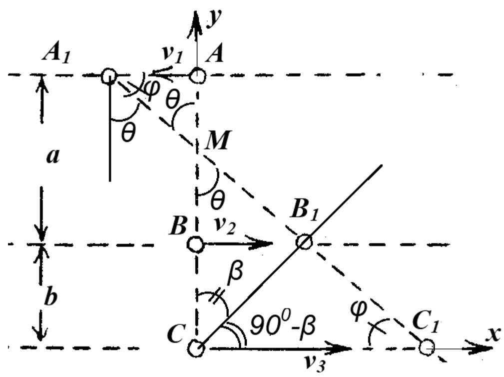

Fig. 9. Case of elongated angle

Considering the notations in Fig. 9, we have:

$$m\left(\measuredangle A_1B_1C\right) + m\left(\measuredangle CB_1C_1\right) = 180^0 \cancel{-\left(\theta+\beta\right)} \cancel{+180^0} \cancel{-180^0} \cancel{+\left(\theta+\beta\right)} = 180^0 \,, \tag{22}$$

Whence points A_1, B_1 and C_1 are collinear and then vectors $\overline{A_1B_1}$ and $\overline{A_1C_1}$ being collinear, their vectorial product is null.

Thus we have:

$$\begin{gathered} \overline{A_1B_1} \times \overline{A_1C_1} = \\ = \left\{\left[v_2 - \left(-v_1\right)\right] t\bar{i} + \left[b - \left(a+b\right)\right]\bar{j}\right\} \times \left\{\left[v_3 - \left(-v_1\right)\right] t\bar{i} + \left[0 - \left(a+b\right)\right]\bar{j}\right\} = 0 \,, \end{gathered} \tag{23}$$

From the development of vectorial product, we have:

$$v_3 a = v_2\left(a+b\right) + v_1 b \,, \tag{24}$$

Equation (24) being identical to equation (13).

5. CONCLUSIONS

Interdisciplinarity is a cooperation between various disciplines of the same curricular area, regarding a certain phenomenon, process, the complexity of which can be demonstrated, explained, solved, only by the action of several factors.

Interdisciplinarity involves approaching the complex contents with the aim of forming a unitary image on a certain subject matter. This implies combining two or several academic disciplines in one single activity. Thus, new knowledge is accumulated in several fields simultaneously.

Mechanics depends on mathematics, and we can realize this by the fact that we cannot solve any problem of mechanics without mathematics.

In the paper the tight connection between the two fundamental disciplines has been shown, mathematics and mechanics.

For the time being, stopping here with the discussion of the problem, we should mention that such a discussion is far from exhausting the multitude of aspects that can be raised. This is a small example supporting the principle of continuity of knowledge, of the fact that the process of knowledge is unlimited, and as in any science, it stays open all the time for acquiring new contributions.

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